

Chapter 5

Dynamic Fracture Analysis of Composites

5.1 INTRODUCTION

5.1.1 Dynamic fracture mechanics

5.1.2 Dynamic fracture mechanics of composites

5.1.3 Dynamic fracture by XFEM

5.2 ANALYTICAL SOLUTIONS FOR NEAR CRACK TIPS IN DYNAMIC STATES

Consider a crack in an anisotropic body with general boundary conditions which is subjected to arbitrary forces, as depicted in Fig. 5.1. In addition to global Cartesian co-ordinates (X_1, X_2) , local Cartesian co-ordinates (x, y) and local polar co-ordinates (r, θ) are defined on the crack-tip.

5.2.1 Analytical solution for a propagating crack in isotropic material

Sih (1968), Sih and Irwin (1970) and Freund (1990) developed the main asymptotic solution for dynamic analysis of propagating cracks in isotropic materials. Later, Nishioka and Atluri (1983, 1984) and Swenson (1986) used the same formulation to present a unified fashion for all fracture modes. Accordingly, the asymptotic displacement field in the local crack tip coordinate system (x, y) in terms of the dynamic stress intensity factors K_I and K_{II} can be written as:

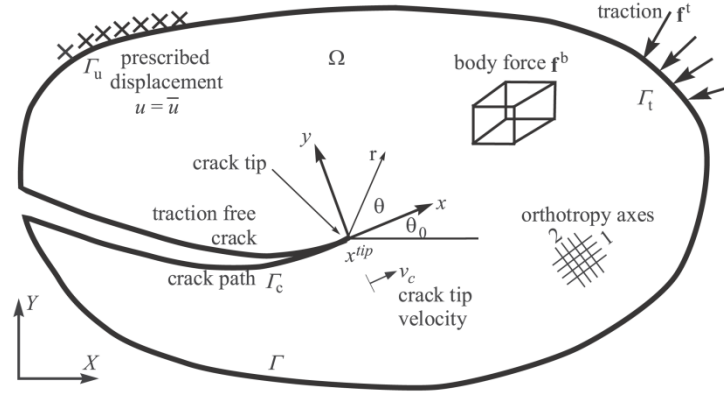


Figure 5.1 An arbitrary orthotropic body subjected to traction $\mathbf{t}$ and arbitrary boundary conditions in a state of elastodynamic equilibrium.

$$u_x = \frac{K_I(1+\beta_s^2)}{\mu D(v_c)} \sqrt{\frac{2}{\pi}} \left\{ \sqrt{r_d} \cos \frac{\theta_d}{2} - \frac{2\beta_d\beta_s}{(1+\beta_s^2)} \sqrt{r_s} \cos \frac{\theta_s}{2} \right\} + \frac{2K_{II}\beta_s}{\mu D(v_c)} \sqrt{\frac{2}{\pi}} \left\{ \sqrt{r_d} \sin \frac{\theta_d}{2} - \frac{(1+\beta_s^2)}{2} \sqrt{r_s} \sin \frac{\theta_s}{2} \right\} \quad (5.1)$$

$$u_y = \frac{K_I(1+\beta_s^2)}{\mu D(v_c)} \sqrt{\frac{2}{\pi}} \left\{ -\beta_d \sqrt{r_d} \sin \frac{\theta_d}{2} + \frac{2\beta_d}{(1+\beta_s^2)} \sqrt{r_s} \sin \frac{\theta_s}{2} \right\} + \frac{2K_{II}\beta_s}{\mu D(v_c)} \sqrt{\frac{2}{\pi}} \left\{ \beta_d \sqrt{r_d} \cos \frac{\theta_d}{2} - \frac{(1+\beta_s^2)}{2\beta_s} \sqrt{r_s} \cos \frac{\theta_s}{2} \right\} \quad (5.2)$$

and the stress components in the same local crack tip coordinates (x, y) are:

$$\sigma_{xx} = \frac{K_I(1+\beta_s^2)}{\sqrt{2\pi} D(v_c)} \left\{ (1+2\beta_d^2-\beta_s^2) r_d^{-\frac{1}{2}} \cos \frac{\theta_d}{2} - \frac{4\beta_d\beta_s}{(1+\beta_s^2)} r_s^{-\frac{1}{2}} \cos \frac{\theta_s}{2} \right\} + \frac{2K_{II}\beta_s}{\sqrt{2\pi} D(v_c)} \left\{ -(1+2\beta_d^2-\beta_s^2) r_d^{-\frac{1}{2}} \sin \frac{\theta_d}{2} + (1+\beta_s^2) r_s^{-\frac{1}{2}} \sin \frac{\theta_s}{2} \right\} \quad (5.3)$$

$$\sigma_{yy} = \frac{K_I(1+\beta_s^2)}{\sqrt{2\pi} D(v_c)} \left\{ -(1+\beta_s^2) r_d^{-\frac{1}{2}} \cos \frac{\theta_d}{2} + \frac{4\beta_d\beta_s}{(1+\beta_s^2)} r_s^{-\frac{1}{2}} \cos \frac{\theta_s}{2} \right\} + \frac{2K_{II}\beta_s}{\sqrt{2\pi} D(v_c)} \left\{ (1+\beta_s^2) r_d^{-\frac{1}{2}} \sin \frac{\theta_d}{2} - (1+\beta_s^2) r_s^{-\frac{1}{2}} \sin \frac{\theta_s}{2} \right\} \quad (5.4)$$

$$\begin{aligned}\sigma_{xy} = & \frac{K_I(1+\beta_s^2)}{\sqrt{2\pi D(v_c)}} \left\{ 2\beta_d r_d^{-\frac{1}{2}} \sin \frac{\theta_d}{2} - 2\beta_d \frac{4\beta_d\beta_s}{(1+\beta_s^2)} r_s^{-\frac{1}{2}} \sin \frac{\theta_s}{2} \right\} \\ & + \frac{2K_{II}\beta_s}{\sqrt{2\pi D(v_c)}} \left\{ 2\beta_d r_d^{-\frac{1}{2}} \cos \frac{\theta_d}{2} - \frac{(1+\beta_s^2)^2}{2\beta_s} r_s^{-\frac{1}{2}} \cos \frac{\theta_s}{2} \right\}\end{aligned}\quad (5.5)$$

σ_{zz} is zero for plane stress problems, whereas it takes the following form for plane strains,

$$\begin{aligned}\sigma_{zz} = & \frac{K_I(1+\beta_s^2)}{\sqrt{2\pi D(v_c)}} \left\{ 2\nu(\beta_d^2 - \beta_s^2) r_d^{-\frac{1}{2}} \cos \frac{\theta_d}{2} \right\} \\ & + \frac{2K_{II}\beta_s}{\sqrt{2\pi D(v_c)}} \left\{ -2\nu(\beta_d^2 - \beta_s^2) r_d^{-\frac{1}{2}} \sin \frac{\theta_d}{2} \right\}\end{aligned}\quad (5.6)$$

v_c is the crack tip speed, and

$$\beta_d^2 = 1 - \frac{v_c^2}{c_d^2} \quad (5.7)$$

$$\beta_s^2 = 1 - \frac{v_c^2}{c_s^2} \quad (5.8)$$

$$D(v_c) = 4\beta_d\beta_s - (1 + \beta_s^2)^2 \quad (5.9)$$

where c_d and c_s are the dilatational and shear wave speeds, respectively,

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (5.10)$$

$$c_s = \sqrt{\frac{\mu}{\rho}} \quad (5.11)$$

and μ and λ are the Lamé coefficients and ρ is the material density. (r_j, θ_j) are defined from the following equation ($i^2 = -1$),

$$r_j e^{i\theta_j} = x + i\beta_j y, \quad j = d, s \quad (5.12)$$

or more explicitly,

$$r_d^2 = x^2 + \beta_d^2 y^2 \quad (5.13)$$

$$r_s^2 = x^2 + \beta_s^2 y^2 \quad (5.14)$$

$$\tan \theta_d = \beta_d \tan \theta \quad (5.15)$$

$$\tan \theta_s = \beta_s \tan \theta \quad (5.16)$$

5.2.2 Asymptotic solution for a stationary crack in orthotropic media

The contracted form of the Hooke's law between linear strain ε_α and stress σ_β in an arbitrary linear elastic homogeneous material can be written as

$$\varepsilon_\alpha = c_{\alpha\beta} \sigma_\beta \quad (\alpha, \beta = 1-6) \quad (5.17)$$

where coefficients $c_{\alpha\beta}$ are defined in (4.8) and (4.12) (see Section 4.2.1).

The displacements and stress fields in the vicinity of the crack-tip, which were elicited by Sih *et al.* (1965) in terms of mode I and mode II stress intensity factors, K_I and K_{II} , respectively, have been defined in Eqs. (4.36)-(4.45) in terms of the roots ($\bar{s}_k = \bar{s}_{kx} + i\bar{s}_{ky}$, $k=1,2$) of the characteristic equation (4.22)

$$c_{11}\bar{s}^4 - 2c_{16}\bar{s}^3 + (2c_{12} + c_{66})\bar{s}^2 - 2c_{26}\bar{s} + c_{22} = 0 \quad (5.18)$$

and variables $\bar{p}_k$ and $\bar{q}_k$, as defined in (4.29) and (4.30), respectively,

$$\bar{p}_k = c_{11}\bar{s}_k^2 + c_{12} - c_{16}\bar{s}_k \quad (5.19)$$

$$\bar{q}_k = c_{12}\bar{s}_k + \frac{c_{22}}{\bar{s}_k} - c_{26} \quad (5.20)$$

5.2.3 Analytical solution for near crack tip of a propagating crack in orthotropic material

The elastodynamic equilibrium equation in the absence of body forces for an orthotropic material, where the material axes are assumed coincident with the local (x, y) coordinate system, can be obtained from the extension of inplane elastostatic equations (4.34) and (4.35): (Piva and Viola, 1988, Brock *et al.*, 2001 and Federici *et al.*, 2001)

$$d_{11} \frac{\partial^2 u_x}{\partial x^2} + d_{66} \frac{\partial^2 u_x}{\partial y^2} + (d_{12} + d_{66}) \frac{\partial^2 u_y}{\partial x \partial y} = \rho \frac{\partial^2 u_x}{\partial t^2} \quad (5.21)$$

$$d_{66} \frac{\partial^2 u_y}{\partial x^2} + d_{22} \frac{\partial^2 u_y}{\partial y^2} + (d_{12} + d_{66}) \frac{\partial^2 u_x}{\partial x \partial y} = \rho \frac{\partial^2 u_y}{\partial t^2} \quad (5.22)$$

Now consider the crack moves with the constant velocity v_c along the local x axis ($x = X - v_c t$ and $y = Y$). The equilibrium equations can be transformed into the moving coordinate system in terms of the new complex variable $z = x + my$ (see Fig. 5.2) using the linear elastic orthotropic constitutive equations (4.31)-(4.33): (Lee *et al.*, 1996)

$$(1 - \rho v_c^2 c_{11}) \frac{\partial \sigma_x}{\partial z} - \rho v_c^2 c_{12} \frac{\partial \sigma_y}{\partial z} + m \frac{\partial \tau_{xy}}{\partial z} = 0 \quad (5.23)$$

$$(-\rho v_c^2 c_{12}) \frac{\partial \sigma_x}{\partial z} + (m^2 - \rho v_c^2 c_{22}) \frac{\partial \sigma_y}{\partial z} + m \frac{\partial \tau_{xy}}{\partial z} = 0 \quad (5.24)$$

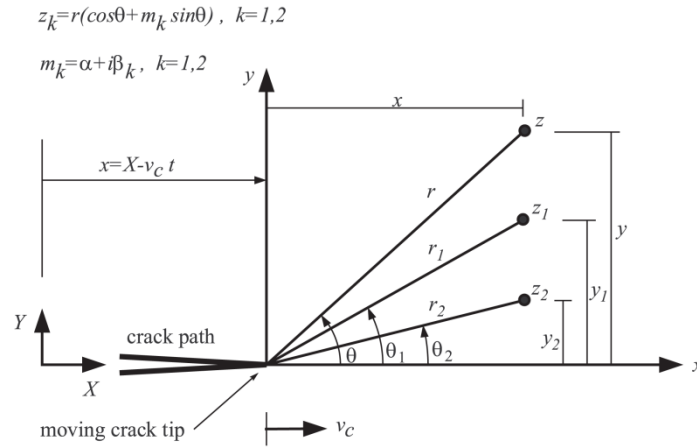


Figure 5.2 Definitions of the dynamic crack propagation under mode I and II loading conditions (Lee *et al.*, 1996).

Lee *et al.* (1996) proposed a method to obtain the solution for a crack which is propagating with constant velocity. The near crack tip displacement fields for a mixed mode dynamic propagating crack can then be expressed as (for details see Lee *et al.*, 1996):

$$u_x = K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{\left[p_2 s_1 \sqrt{\cos \theta + m_2 \sin \theta} - p_1 s_2 \sqrt{\cos \theta + m_1 \sin \theta} \right]}{\left[1 + \rho v_c^2 (c_{12} - c_{11}) \right] (s_1 - s_2)} \right\} + K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{\left[p_2 \sqrt{\cos \theta + m_2 \sin \theta} - p_1 \sqrt{\cos \theta + m_1 \sin \theta} \right]}{(s_1 - s_2)} \right\} \quad (5.25)$$

$$\begin{aligned}
u_y = & K_I \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{[q_2 s_1 \sqrt{\cos \theta + m_2 \sin \theta} - q_1 s_2 \sqrt{\cos \theta + m_1 \sin \theta}]}{[1 + \rho v_c^2 (c_{12} - c_{11})] (s_1 - s_2)} \right\} \\
& + K_{II} \sqrt{\frac{2r}{\pi}} \operatorname{Re} \left\{ \frac{[q_2 \sqrt{\cos \theta + m_2 \sin \theta} - q_1 \sqrt{\cos \theta + m_1 \sin \theta}]}{(s_1 - s_2)} \right\}
\end{aligned} \quad (5.36)$$

and the near crack tip stress components are described as:

$$\begin{aligned}
\sigma_x = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{\left[\frac{[1 + m_2^2 + \rho v_c^2 (c_{12} - c_{22})] s_1}{\sqrt{\cos \theta + m_2 \sin \theta}} - \frac{[1 + m_1^2 + \rho v_c^2 (c_{12} - c_{22})] s_2}{\sqrt{\cos \theta + m_1 \sin \theta}} \right]}{[1 + \rho v_c^2 (c_{12} - c_{11})] (s_1 - s_2)} \right\} \\
& + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{\left[\frac{[1 + m_2^2 + \rho v_c^2 (c_{12} - c_{22})]}{\sqrt{\cos \theta + m_2 \sin \theta}} - \frac{[1 + m_1^2 + \rho v_c^2 (c_{12} - c_{22})]}{\sqrt{\cos \theta + m_1 \sin \theta}} \right]}{(s_1 - s_2)} \right\}
\end{aligned} \quad (5.27)$$

$$\begin{aligned}
\sigma_y = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{(s_1 - s_2)} \left[\frac{s_1}{\sqrt{\cos \theta + m_2 \sin \theta}} - \frac{s_2}{\sqrt{\cos \theta + m_1 \sin \theta}} \right] \right\} \\
& + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{[1 + \rho v_c^2 (c_{12} - c_{22})]}{(s_1 - s_2)} \left[\frac{1}{\sqrt{\cos \theta + m_2 \sin \theta}} - \frac{1}{\sqrt{\cos \theta + m_1 \sin \theta}} \right] \right\}
\end{aligned} \quad (5.28)$$

$$\begin{aligned}
\tau_{xy} = & \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{s_1 s_2 \left[\frac{s_1}{\sqrt{\cos \theta + m_1 \sin \theta}} - \frac{s_2}{\sqrt{\cos \theta + m_2 \sin \theta}} \right]}{[1 + \rho v_c^2 (c_{12} - c_{22})] (s_1 - s_2)} \right\} \\
& + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{(s_1 - s_2)} \left[\frac{s_1}{\sqrt{\cos \theta + m_1 \sin \theta}} - \frac{s_2}{\sqrt{\cos \theta + m_2 \sin \theta}} \right] \right\}
\end{aligned} \quad (5.29)$$

where p_i , q_i and s_i are defined as:

$$p_1 = c_{11} [1 + m_1^2 + \rho v_c^2 (c_{12} - c_{22})] + c_{12} [1 + \rho v_c^2 (c_{12} - c_{11})] \quad (5.30)$$

$$p_2 = c_{11} [1 + m_2^2 + \rho v_c^2 (c_{12} - c_{22})] + c_{12} [1 + \rho v_c^2 (c_{12} - c_{11})] \quad (5.31)$$

$$q_1 = \frac{c_{12} [1 + m_1^2 + \rho v_c^2 (c_{12} - c_{22})] + c_{22} [1 + \rho v_c^2 (c_{12} - c_{11})]}{m_1} \quad (5.32)$$

$$q_2 = \frac{c_{12} \left[1 + m_2^2 + \rho v_c^2 (c_{12} - c_{22}) \right] + c_{22} \left[1 + \rho v_c^2 (c_{12} - c_{11}) \right]}{m_2} \quad (5.33)$$

$$s_1 = m_1 - \rho v_c^2 c_{22} / m_1 - m_1 \rho v_c^2 c_{11} + \rho^2 v_c^4 (c_{11} c_{22} - c_{12}^2) / m_1 \quad (5.34)$$

$$s_2 = m_2 - \rho v_c^2 c_{22} / m_2 - m_2 \rho v_c^2 c_{11} + \rho^2 v_c^4 (c_{11} c_{22} - c_{12}^2) / m_2 \quad (5.35)$$

Coefficients m_j ($j=1,2$) are calculated from the following characteristic equation:

$$(m^4 + 2B_{12}m^2 + B_{66}) = 0 \quad (5.36)$$

where B_{12} and B_{66} are defined as:

$$2B_{12} = \frac{1}{c_{11}} \left[2c_{12} + c_{66} + \rho v_c^2 (c_{12}^2 - c_{11}c_{66} - c_{11}c_{22}) \right] \quad (5.37)$$

$$B_{66} = \frac{1}{c_{11}} \left\{ c_{22} + \rho v_c^2 \left[c_{12}^2 - c_{22}c_{66} - c_{11}c_{22} + \rho v_c^2 c_{66} (c_{11}c_{22} - c_{12}^2) \right] \right\} \quad (5.38)$$

5.3 DYNAMIC STRESS INTENSITY FACTORS

Several methods have been proposed for computation of stress intensity factors. They include formulations proposed by Nishioka and Atluri (1984), Freund (1989), Dongye and Ting (1989), Aliabadi and Sollero (1998), Wu (2000) and Kim and Paulino (2002c), etc.

5.3.1 Stationary and moving crack dynamic stress intensity factors

Dynamic stress intensity factors, which play an important role in dynamic fracture problems, are used to calculate the maximum hoop stress intensity factor and to evaluate dynamic crack propagation properties.

Freund (1972) obtained the ideal mode I and II dynamic stress intensity factors for a stationary crack subjected to normal and shear incident plane stress pulse with a jump in stress of magnitude σ_0 and τ_0 , respectively, (Freund, 1989)

$$K_I(0,t) = \frac{2\sigma_0}{1-\nu} \sqrt{\frac{c_d t (1-2\nu)}{\pi}} \quad (5.39)$$

$$K_{II}(0,t) = 2\tau_0 \sqrt{\frac{2c_s t}{\pi(1-\nu)}} \quad (5.40)$$

Eqs. (5.39) and (5.40) can be extended to moving crack tip problems by (Freund 1972)

$$K_I(v_c, t) = f_I^d(v_c) K_I(0, t) \quad (5.41)$$

$$K_{II}(v_c, t) = f_{II}^d(v_c) K_{II}(0, t) \quad (5.42)$$

where f_I^d and f_{II}^d are universal functions of the crack tip speed v_c , which can be approximated for isotropic problems by (Freund, 1972 and 1989)

$$f_I^d(v_c) = \frac{1 - \frac{v_c}{c_R}}{\sqrt{1 - \frac{v_c}{c_d}}} \quad (5.43)$$

$$f_{II}^d(v_c) = \frac{1 - \frac{v_c}{c_R}}{\sqrt{1 - \frac{v_c}{c_s}}} \quad (5.44)$$

where c_R is the Rayleigh wave speed, which is the positive root of the following equation (Destrade, 2007),

$$c_R^4(d_{11} - c_R^2) = \frac{\rho^2 d_{22}}{d_{66}} (d_0 - c_R^2)^2 (d_{66} - c_R^2) \quad (5.45)$$

where

$$d_0 = d_{11} - \frac{d_{12}^2}{d_{22}} \quad (5.46)$$

Equation (5.45) for isotropic problems is reduced to (Pichugin *et al.*, 2008)

$$c_{Rs}^6 - 8c_{Rs}^4 + \frac{8(2-\nu)}{1-\nu} c_{Rs}^2 - \frac{8}{1-\nu} = 0, \quad c_{Rs} = \frac{c_R}{c_s} \quad (5.47)$$

with a final approximate solution for c_R by Bower (2012),

$$c_R = c_s \left[0.875 - 0.2\nu - 0.05(\nu + 0.25)^3 \right] \quad (5.48)$$

5.3.2 Dynamic fracture criteria

The dynamic crack propagation law should realistically be chosen based upon the type of simulated or experimented material. Despite the fact that simulation of dynamic brittle

crack propagation remains a difficult challenge, numerical modeling of fracture phenomenon can be performed by relatively simple concepts. First, a criterion is required to describe the state of stability of an existing crack. Then, a rule is needed to anticipate the propagation direction, if the crack becomes unstable. Finally, another equation is necessary to specify the crack propagation speed (Gregoire *et al.*, 2007).

The main idea is to compare the stress intensity factor with the dynamic crack initiation toughness to evaluate the crack stability/instability. The dynamic crack initiation toughness is a material property and can be obtained by experiment. If the crack violates such a stability criterion, it starts to propagate and the direction of propagation is evaluated from a propagation criterion such as the maximum hoop stress (Maigre and Rittel, 1993).

During the crack propagation, the crack speed adapts its value in a way to make the maximum hoop stress intensity factor become equal to the dynamic crack growth toughness. To evaluate the dynamic crack growth toughness, Kanninen and Popelar (1985) proposed to replace the quasi-static toughness by the dynamic crack initiation toughness. In this method, the dynamic crack growth toughness is considered to be (Gregoire *et al.*, 2007):

$$\begin{cases} K_{tt} < K_{Ic}^d & : \text{stable crack} \\ K_{tt}(t,0) = K_{Ic}^d & : \text{initiation at } \theta = \theta_c \\ K_{tt}(t, v_c) \geq K_{Ic}^d \Rightarrow K_{tt}(t, v_c) = K_{Ic}^D(v_c) & : \text{propagation} \end{cases} \quad (5.49)$$

where K_{tt} is the dynamic maximum hoop stress intensity factor, K_{Ic}^d is the dynamic crack initiation toughness and K_{Ic}^D is the dynamic crack growth (propagation) toughness (Gregoire *et al.*, 2007 and Elguedj *et al.*, 2009)

$$K_{Ic}^D(v_c) = \frac{K_{Ic}^d}{\left(1 - \frac{v_c}{c_R}\right)} \quad (5.50)$$

v_c can be elicited from (5.49) and (5.50) on propagation (Freund (1972,1990), Freund and Douglass (1982) and Rosakis and Freund (1982)):

$$v_c = c_R \left(1 - \frac{K_{Ic}^d}{K_{tt}}\right) \quad (5.51)$$

Alternatively, Menouillard *et al.* (2010) have used the following approximation

$$v_c = c_R \left(1 - \left(\frac{K_{Ic}^d}{K_{tt}}\right)^2\right) \quad (5.52)$$

The maximum hoop/circumferential stress criterion (Section 4.4.4.1) or the minimum strain energy density criterion (Section 4.4.4.2) can be used to determine the propagation angle.

5.3.3 J integral for dynamic problems

The analytical form of dynamic J integral (J_k^d) in the global coordinate system $k = X, Y$, developed by Nishioka and Alturi (1984), for an infinitesimally small internal contour Γ_1 can be written as:

$$J_k^d = \int_{\Gamma+\Gamma_c} ((w_s + w_d)n_k - t_i u_{i,k}) d\Gamma + \int_{A-A_1} ((\rho \ddot{u}_i - f_i^b)u_{i,k} - \rho \dot{u}_{i,k} \dot{u}_i) dA \quad (5.53)$$

where u_i , t_i and f_i^b denote the displacement, traction and body force, respectively n_k is the component of normal outward vector $\mathbf{n} = (n_X, n_Y)$ and the integral paths Γ_1 , Γ , and Γ_c denote near-field, far-field and crack surface paths, respectively. A_1 and A_c are the regions surrounded by Γ and Γ_1 , respectively (Fig. 5.3). The strain energy density w_s and the kinetic energy density w_d are defined as,

$$w_s = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \quad (5.54)$$

$$w_d = \frac{1}{2} \rho \dot{u}_i \dot{u}_i \quad (5.55)$$

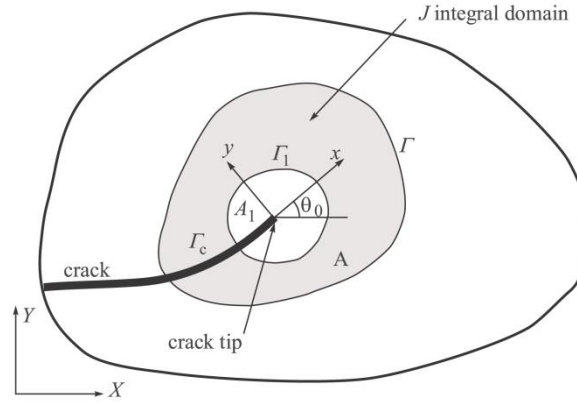


Figure 5.3 Definitions of crack path contours and the local crack tip coordinates for evaluation of the J integral.

5.3.4. Domain integral for orthotropic media

Kim and Paulino (2003d) proposed an equivalent form of Eq. (5.53), using the divergence theorem and some additional assumptions for homogeneous materials, which is better suited for the finite element method (in the global system $k = X, Y$),

$$J_k^d = \int_A (\sigma_{ij} u_{i,k} - (w_s + w_d)) q_{,k} dA + \int_A ((\rho \ddot{u}_i - f_i^b) u_{i,k} - \rho \dot{u}_{i,k} \dot{u}_i) q dA \quad (5.56)$$

where q is a smooth function, assumed $q = 1$ inside A_1 and varying linearly from 1 at Γ_1 near the crack-tip to $q = 0$ at the exterior boundary Γ , as discussed in Section 2.7.4. Noting that the value of q is constant near the crack-tip area, the gradient of q vanishes inside A_1 .

It should be noted that in a general transient analysis, potential oscillations of $\ddot{u}_i$ may affect the quality of J integral computations through Eq. (5.53) or (5.54). Fortunately, this is not the case for dynamic analysis of stationary cracks. Among several investigations on this issue, Gregoire *et al.* (2007) have performed an extensive study on stability of dynamic fracture mechanics and conservation of energy. They showed that utilising the Newmark's implicit mean acceleration approach combined with enriching the cracked elements will provide sufficient accuracy for finding u_i , $\dot{u}_i$, $\ddot{u}_i$ and so the J integral.

5.3.5 Interaction integral

Evaluation of mixed mode J integral and associated stress intensity factors in XFEM is usually performed by means of interaction integral methods. Since the interaction integral method requires several auxiliary field components, equations of dynamic near-tip displacement and stress fields in orthotropic materials have to be derived in order to extract the required auxiliary displacement and stress fields. Moreover, more complex computations are required to calculate 'auxiliary velocity' and 'auxiliary strain velocity' fields especially in orthotropic media. These require a substantial theoretical and simulation experience and cost.

The general form of dynamic interaction integral for a moving crack can be defined as (Suo and Combescure, 1992, Attigui and Petit, 1997 and Rethore *et al.*, 2004)

$$M^d(t) = \frac{1}{2} \int_{\Gamma} \left[(\sigma_{ij,1}^{aux} u_i - \sigma_{ij} u_{i,1}^{aux}) n_j - \rho \dot{u}_i \dot{u}_i^{aux} n_i - v_c \rho (\dot{u}_i \dot{u}_{i,1}^{aux} + \dot{u}_i^{aux} \dot{u}_{i,1}) n_1 \right] d\Gamma + \frac{1}{2} \frac{d}{dt} \int_A \left[\rho (\dot{u}_i \dot{u}_{i,1}^{aux} + \dot{u}_i^{aux} \dot{u}_{i,1}) \right] dA \quad (5.57)$$

Neglecting the effect of crack tip velocity, the equivalent domain form of the dynamic interaction integral can be written as (Rethore *et al.*, 2005c, Gregoire *et al.*, 2007, Combescure *et al.*, 2008 and Elguedj *et al.*, 2009)

$$M^d(t) = - \int_A \left[(\sigma_{m1}^{aux} u_{m,1} - \rho \dot{u}_1 \dot{u}_1^{aux}) \delta_{kj} - (\sigma_{ij}^{aux} u_{i,k} + \sigma_{ij} u_{i,k}^{aux}) \right] q_{k,j} dA + 2 \int_A \left[\sigma_{ij,j}^{aux} u_{i,k} + \rho \ddot{u}_i u_{i,k}^{aux} + \rho \dot{u}_{i,k} \dot{u}_i^{aux} + \rho \dot{u}_i \dot{u}_{i,k}^{aux} \right] q_k dA \quad (5.58)$$

Evaluation of the dynamic stress intensity factors K_I and K_{II} from the M^d integral for isotropic problems can be performed from (Menouillard *et al.*, 2010).

$$M^d(t) = \frac{2}{E'} \left[f_I(v_c) K_I K_I^{aux} + f_{II}(v_c) K_{II} K_{II}^{aux} \right] \quad (5.59)$$

by setting $K_I^{aux} = 1, K_{II}^{aux} = 0$ and $K_I^{aux} = 0, K_{II}^{aux} = 1$, respectively. Functions $f_I(v_c)$ and $f_{II}(v_c)$ are defined as

$$f_I(v_c) = \frac{4\beta_d(1-\beta_s^2)}{(\kappa+1)D(v_c)} \quad (5.60)$$

$$f_{II}(v_c) = \frac{4\beta_s(1-\beta_s^2)}{(\kappa+1)D(v_c)} \quad (5.61)$$

It should be noted that evaluation of the interaction integral requires careful attention as the main fields are usually obtained from the finite element solution in a crack-independent global or local coordinate system, while the auxiliary fields are defined in the local crack tip polar coordinate system. Therefore, necessary transformations are required to use a unified coordinate system.

5.3.6. Crack-axis component of the dynamic J integral

Computation of J_X^d from (5.56) is usually less affected by the way the crack tip is modelled, whereas J_Y^d is sensitive to modelling of the crack tip due to the singularity of $w_s + w_d$ in evaluation of J_Y^d . To overcome this difficulty in the direct method, Nishioka *et al.* (2001), adopted the component separation method.

The local crack-axis (tangential) components (J_l^d) of the dynamic J integral, which corresponds to the rate of change in the potential energy per unit crack extension (G), can be evaluated by the following coordinate transformation (Kim and Paulino, 2003d and Nishioka *et al.*, 2001):

$$J_l^d = \alpha_{lk}(\theta_0) J_k^d, \quad k = X, Y \text{ and } l = x, y \quad (5.62)$$

where α_{lk} is the coordinate transformation tensor and θ_0 is the crack angle with horizontal axis. Wu (2000) showed that the dynamic energy release rate G can be related to the instantaneous stress intensity factors for an elastodynamic self-similar propagating crack with the velocity v_c ,

$$G = J_x^d = J_X^d \cos \theta_0 + J_Y^d \sin \theta_0 \quad (5.63)$$

5.3.6.1 Isotropic formulation

The displacements of crack edges ($\mathbf{u}^+, \mathbf{u}^-$) for each loading mode can be calculated by transforming the numerically computed displacement field into the local polar crack tip coordinates (x, y) ,

$$\delta_I = u_y^+ - u_y^- \quad (5.64)$$

$$\delta_{II} = u_x^+ - u_x^- \quad (5.65)$$

Nishioka and Atluri (1980a,b, 1984) illustrated that while G can be computed from (5.63) for linear and nonlinear elastic homogeneous or inhomogeneous materials, the stress intensity factors for homogenous linear elastic materials can be obtained from the following equations,

$$K_I = \delta_I \left\{ \frac{(2\mu\beta_s D(v_c)) J_x^d}{\beta_d(1-\beta_s^2) [\delta_I^2 \beta_s + \delta_{II}^2 \beta_d]} \right\}^{\frac{1}{2}} \quad (5.66)$$

$$K_{II} = \delta_{II} \left\{ \frac{(2\mu\beta_d D(v_c)) J_x^d}{\beta_s(1-\beta_s^2) [\delta_I^2 \beta_s + \delta_{II}^2 \beta_d]} \right\}^{\frac{1}{2}} \quad (5.67)$$

5.3.6.2 Orthotropic formulation

Now, the extension of the method to orthotropic media is explained. Beginning with the definition of G (Kim and Paulino, 2003d)

$$G = \frac{1}{2} \mathbf{K}^T \mathbf{L}^{-1}(v_c) \mathbf{K} \quad (5.68)$$

where $\mathbf{L}(v_c)$ has been presented by Dongye and Ting (1989) for orthotropic materials with the symmetry planes coinciding with the coordinate planes, and $\mathbf{K}$ includes the stress intensity factors.

The nonzero components of $\mathbf{L}(v_c)$ are:

$$\sqrt{d_{66}^d d_{22}} L_{11}(v_c) = \sqrt{d_{11}^d d_{66}} L_{22}(v_c) = \Omega \Psi^{-\frac{1}{2}} \quad (5.69)$$

where $d_{ij}^d = d_{ij} - \rho v_c^2 \delta_{ij}$, d_{ij} are the constitutive coefficients defined in (4.10) and

$$\Omega = (d_{11}^d d_{22} - d_{12}^2) \sqrt{d_{66}^d d_{66}} - \rho v_c^2 d_{66} \sqrt{d_{11}^d d_{22}} \quad (5.70)$$

$$\Psi = (\sqrt{d_{66}^d d_{66}} + \sqrt{d_{11}^d d_{22}})^2 - (d_{12} + d_{66})^2 \quad (5.71)$$

To accurately evaluate the in-plane mixed-mode stress intensity factors from the dynamic J integral, the component separation method, proposed by Aliabadi and Sollero

(1998), is implemented. The method is based on the following relationship between the dynamic stress intensity factors and the relative sliding and opening displacements of the crack face:

$$\begin{Bmatrix} \delta_I \\ \delta_{II} \end{Bmatrix} = \sqrt{\left(\frac{8r}{\pi}\right)} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} \quad (5.72)$$

with

$$\begin{aligned} A_{11} &= \text{Im} \left(\frac{\bar{s}_2 \bar{p}_1 - \bar{s}_1 \bar{p}_2}{\bar{s}_1 - \bar{s}_2} \right), \quad A_{12} = \text{Im} \left(\frac{\bar{p}_1 - \bar{p}_2}{\bar{s}_1 - \bar{s}_2} \right) \\ A_{21} &= \text{Im} \left(\frac{\bar{s}_2 \bar{q}_1 - \bar{s}_1 \bar{q}_2}{\bar{s}_1 - \bar{s}_2} \right), \quad A_{22} = \text{Im} \left(\frac{\bar{q}_1 - \bar{q}_2}{\bar{s}_1 - \bar{s}_2} \right) \end{aligned} \quad (5.73)$$

where $\bar{s}_k$, $\bar{p}_k$ and $\bar{q}_k$ are defined in Eqs. (5.18)-(5-20). The ratio of opening to sliding displacements, R_δ , can be written as:

$$R_\delta = \frac{\delta_{II}}{\delta_I} = \frac{A_{21}K_I + A_{22}K_{II}}{A_{11}K_I + A_{12}K_{II}} \quad (5.74)$$

Therefore, the ratio of dynamic stress intensity factors, R_K , is obtained as:

$$R_K = \frac{K_I}{K_{II}} = \frac{R_\delta A_{21} - A_{22}}{A_{21} + R_\delta A_{11}} \quad (5.75)$$

Substitution for K_I from Eq. (5.75) into Eq. (5.74) leads to the following relationship for K_{II} :

$$K_{II} = \sqrt{\frac{2G}{L_{11}(v_c)R_K^2 + L_{22}(v_c)}} \quad (5.76)$$

5.3.7 Field decomposition technique

Decomposition of displacement and stress fields into two independent symmetric and anti-symmetric modes, as described in Section 2.5.5, can also be used in combination with the analytical solutions of Section 5.2.3 to compute the stress intensity factors. This is considered as a direct approach of determining the stress intensity factors either on a locally symmetric mesh around the crack, or based on symmetric points with respect to the crack direction on any arbitrarily generated mesh. For further details see Nishioka and Atluri, 1984.

5.4 DYNAMIC XFEM

5.4.1 Dynamic equations of motion

Consider a body Ω with an initial traction-free crack in the state of dynamic equilibrium, as depicted in Fig. 5.1. The fundamental elastodynamic equation can be expressed as:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}^b = \rho \ddot{\mathbf{u}} \quad (5.77)$$

with the following boundary conditions:

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) \quad \text{on } \Gamma_u \quad (5.78)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}^t \quad \text{on } \Gamma_t \quad (5.79)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_c \quad (5.80)$$

and initial conditions:

$$\mathbf{u}(\mathbf{x}, t = 0) = \bar{\mathbf{u}}(0) \quad (5.81)$$

$$\dot{\mathbf{u}}(\mathbf{x}, t = 0) = \bar{\dot{\mathbf{u}}}(0) \quad (5.82)$$

where Γ_t , Γ_u and Γ_c are traction, displacement and crack boundaries, respectively; $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{f}^b$ and $\mathbf{f}^t$ are the body force and external traction vectors, respectively.

The variational formulation of the initial/boundary value problem of Eq. (5.77) can be written as:

$$\int_{\Omega} \rho \ddot{\mathbf{u}} \cdot \delta \mathbf{u} \, d\Omega + \int_{\Omega} \boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon} \, d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma} \mathbf{f}^t \cdot \delta \mathbf{u} \, d\Gamma \quad (5.83)$$

5.4.2 XFEM discretisation

Assuming a discontinuity (a crack) within an independent finite element mesh, the displacement field for any typical point $\mathbf{x}$ inside the domain can be written in terms of the classical finite element approximation and the XFEM enriched fields (see Section 3.4.3):

$$\begin{aligned}
\mathbf{u}^h(\mathbf{x}) &= \mathbf{u}(\mathbf{x}) + \mathbf{u}^H(\mathbf{x}) + \mathbf{u}^{tip}(\mathbf{x}) \\
&= \left[\sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \right] + \left[\sum_{h=1}^{mh} N_h(\mathbf{x}) (H(x) - H(x_h)) \mathbf{a}_h \right] \\
&\quad + \left[\sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} [F_l(x) - F_l(x_k)] \mathbf{b}_k^l \right) \right]
\end{aligned} \tag{5.84}$$

where N_i is the shape function associated with the node i , $H(\mathbf{x})$ is the Heaviside enrichment function and $F_l(x)$ are crack tip enrichment functions. $\mathbf{u}_j$ are the displacement degrees of freedom, and $\mathbf{a}_h$ and $\mathbf{b}_k^l$ are additional Heaviside and tip enrichment degrees of freedom, respectively.

The discretised form of Eq. (5.83) using the XFEM approximation (5.84) can be written as:

$$\mathbf{M} \ddot{\mathbf{u}}^h + \mathbf{K} \mathbf{u}^h = \mathbf{f} \tag{5.85}$$

where $\mathbf{u}^h$ and $\ddot{\mathbf{u}}^h$ denote the vector of nodal parameters (displacements $\mathbf{u}$, Heaviside and crack tip enrichment degrees of freedom $\mathbf{a}$ and $\mathbf{b}$, respectively) and its second time derivative, respectively:

$$\mathbf{u}^h = \{\mathbf{u}, \mathbf{a}, \mathbf{b}\}^T \tag{5.86}$$

$$\ddot{\mathbf{u}}^h = \{\ddot{\mathbf{u}}, \ddot{\mathbf{a}}, \ddot{\mathbf{b}}\}^T \tag{5.87}$$

The stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$ and the external load vector $\mathbf{f}$ are defined as:

$$\mathbf{K}_{ij}^e = \begin{bmatrix} \mathbf{K}_{ij}^{uu} & \mathbf{K}_{ij}^{ua} & \mathbf{K}_{ij}^{ub} \\ \mathbf{K}_{ij}^{au} & \mathbf{K}_{ij}^{aa} & \mathbf{K}_{ij}^{ab} \\ \mathbf{K}_{ij}^{bu} & \mathbf{K}_{ij}^{ba} & \mathbf{K}_{ij}^{bb} \end{bmatrix} \tag{5.88}$$

$$\mathbf{M}_{ij}^e = \begin{bmatrix} \mathbf{M}_{ij}^{uu} & \mathbf{M}_{ij}^{ua} & \mathbf{M}_{ij}^{ub} \\ \mathbf{M}_{ij}^{au} & \mathbf{M}_{ij}^{aa} & \mathbf{M}_{ij}^{ab} \\ \mathbf{M}_{ij}^{bu} & \mathbf{M}_{ij}^{ba} & \mathbf{M}_{ij}^{bb} \end{bmatrix} \tag{5.89}$$

$$\mathbf{f}_i = \{\mathbf{f}_i^u, \mathbf{f}_i^a, \mathbf{f}_i^b\}^T \tag{5.90}$$

The stiffness components $\mathbf{K}_{ij}^{rs}(r, s = \mathbf{u}, \mathbf{a}, \mathbf{b})$ include the classical FEM ($\mathbf{uu}$), Heaviside enrichment ($\mathbf{aa}$), orthotropic crack tip enrichment ($\mathbf{bb}$) and the coupled parts of XFEM approximation:

$$\mathbf{K}_{ij}^{rs} = \int_{\Omega^e} (\mathbf{B}_i^r)^T \mathbf{D} \mathbf{B}_j^s d\Omega \quad (r, s = \mathbf{u}, \mathbf{a}, \mathbf{b}) \quad (5.91)$$

where $\mathbf{B}_i^{\mathbf{u}}$, $\mathbf{B}_i^{\mathbf{a}}$ and $\mathbf{B}_i^{\mathbf{b}}$ are matrices of derivatives of shape functions, defined in equations (3.49), (3.50) and (3.51), respectively.

Classical and enrichment components of the consistent mass matrix can be expressed as:

$$\mathbf{M}_{ij}^{\mathbf{uu}} = \int_{\Omega} \rho N_i N_j d\Omega \quad (5.92)$$

$$\mathbf{M}_{ij}^{\mathbf{aa}} = \int_{\Omega} \rho [N_i (H(\xi) - H(\xi_i))] [N_j (H(\xi) - H(\xi_j))] d\Omega \quad (5.93)$$

$$\mathbf{M}_{ij}^{\mathbf{bb}} = \int_{\Omega} \rho [N_i (F_{\alpha} - F_{\alpha i})] [N_j (F_{\alpha} - F_{\alpha j})] d\Omega \quad (\alpha = 1, 2, 3 \text{ and } 4) \quad (5.94)$$

$$\mathbf{M}_{ij}^{\mathbf{ua}} = \mathbf{M}_{ij}^{\mathbf{au}} = \int_{\Omega} \rho N_i [N_j (H(\xi) - H(\xi_j))] d\Omega \quad (5.95)$$

$$\mathbf{M}_{ij}^{\mathbf{ub}} = \mathbf{M}_{ij}^{\mathbf{bu}} = \int_{\Omega} \rho N_i [N_j (F_{\alpha} - F_{\alpha j})] d\Omega \quad (\alpha = 1, 2, 3 \text{ and } 4) \quad (5.96)$$

$$\mathbf{M}_{ij}^{\mathbf{ab}} = \int_{\Omega} \rho [N_i (H(\xi) - H(\xi_i))] [N_j (F_{\alpha} - F_{\alpha j})] d\Omega \quad (\alpha = 1, 2, 3 \text{ and } 4) \quad (5.97)$$

The diagonal mass matrix can be computed by the lumping technique proposed by Menouillard *et al.* (2008).

Finally, the force vectors associated with the classical and enrichment degrees of freedom are defined in (3.45)-(3.47):

$$\mathbf{f}_i^{\mathbf{u}} = \int_{\Gamma_t} N_i \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i \mathbf{f}^b d\Omega \quad (5.98)$$

$$\mathbf{f}_i^{\mathbf{a}} = \int_{\Gamma_t} N_i [H(\xi) - H(\xi_i)] \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i [H(\xi) - H(\xi_i)] \mathbf{f}^b d\Omega \quad (5.99)$$

$$\mathbf{f}_i^{\mathbf{ba}} = \int_{\Gamma_t} N_i (F_{\alpha} - F_{\alpha i}) \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i (F_{\alpha} - F_{\alpha i}) \mathbf{f}^b d\Omega \quad (\alpha = 1, 2, 3 \text{ and } 4) \quad (5.100)$$

5.4.3 XFEM Enrichment Functions

5.4.3.1 Orthotropic crack-tip functions for stationary cracks

The following orthotropic crack-tip enrichment functions, derived by Asadpoure and Mohammadi (2007), can be adopted for stationary cracks:

$$\{F_I(r, \theta)\}_{I=1}^4 = \left\{ \sqrt{r} \cos \frac{\bar{\theta}_1}{2} \sqrt{\bar{g}_1(\theta)}, \sqrt{r} \cos \frac{\bar{\theta}_2}{2} \sqrt{\bar{g}_2(\theta)}, \right. \\ \left. \sqrt{r} \sin \frac{\bar{\theta}_1}{2} \sqrt{\bar{g}_1(\theta)}, \sqrt{r} \sin \frac{\bar{\theta}_2}{2} \sqrt{\bar{g}_2(\theta)} \right\} \quad (5.101)$$

where r and θ define the crack-tip local polar coordinates, and $\bar{g}_k(\theta)$ and $\bar{\theta}_k$, ($k=1,2$) are defined as:

$$\bar{g}_k(\theta) = \sqrt{(\cos \theta + \bar{s}_{kx} \sin \theta)^2 + (\bar{s}_{ky} \sin \theta)^2} \quad (5.102)$$

$$\bar{\theta}_k = \arctg \left(\frac{\bar{s}_{ky} \sin \theta}{\cos \theta + \bar{s}_{kx} \sin \theta} \right) \quad (5.103)$$

and $\bar{s}_k = \bar{s}_{kx} + i\bar{s}_{ky}$ are the roots of Eq. (5.18).

In addition to static and quasi-static problems, the static orthotropic enrichments (5.101) have already been used for analysis of stationary and moving cracks subjected to dynamic loadings by Motamedi and Mohammadi (2010a,b).

5.4.3.2 Orthotropic crack-tip functions for moving cracks

In order to derive crack-tip enrichment functions from the analytical solutions (5.25) and (5.26), the following polar transformation is adopted:

$$Z_k^{aux} = r_k e^{i\theta_k} = r(\cos \theta + m_k \sin \theta), \quad k=1, 2 \quad (5.104)$$

where r_k and $g_k(\theta)$ are defined as:

$$r_k = r g_k(\theta), \quad k=1, 2 \quad (5.105)$$

$$g_k(\theta) = \sqrt{(\cos \theta + m_{kx} \sin \theta)^2 + (m_{ky} \sin \theta)^2}, \quad k=1, 2 \quad (5.106)$$

$$\theta_k = \arctg \left(\frac{m_{ky} \sin \theta}{\cos \theta + m_{kx} \sin \theta} \right), \quad k=1, 2 \quad (5.107)$$

and (r, θ) are considered from the crack-tip position, as depicted in Figure 5.1. m_{kx} and m_{ky} are the real and imaginary parts of m_k (roots of Eq. (5.36)):

$$m_{kx} = \text{Re}(m_k), \quad k=1, 2 \quad (5.108)$$

$$m_{ky} = \text{Im}(m_k), \quad k=1, 2 \quad (5.109)$$

Thereafter, the imaginary and real parts of the main displacement fields can be rewritten as (Figure 5.2):

$$\sqrt{(\cos \theta + m_k \sin \theta)}, (k=1, 2) \quad (5.110)$$

$$\operatorname{Im}\left(\sqrt{Z_k^{aux}}\right) = r^{1/2} \sqrt{g_k(\theta)} \sin \frac{\theta_k}{2}, \quad k=1, 2 \quad (5.111)$$

$$\operatorname{Re}\left(\sqrt{Z_k^{aux}}\right) = r^{1/2} \sqrt{g_k(\theta)} \cos \frac{\theta_k}{2}, \quad k=1, 2 \quad (5.112)$$

Finally, the following dynamic orthotropic enrichment functions can be derived for moving crack tip problems:

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \cos \frac{\theta_2}{2} \sqrt{g_2(\theta)}, \right. \\ \left. \sqrt{r} \sin \frac{\theta_1}{2} \sqrt{g_1(\theta)}, \sqrt{r} \sin \frac{\theta_2}{2} \sqrt{g_2(\theta)} \right\} \quad (5.113)$$

where $g_k(\theta)$ and θ_k , ($k=1,2$) are defined in equations (5.106) and (5.107).

The present formulation can be used for modeling all groups of composites and is in interestingly similar form as those presented in Asadpoure and Mohammadi (2007) but with different definitions for parameters. These functions, in fact, can be degenerated to static orthotropic enrichment functions developed by Asadpoure and Mohammadi (2007) if dynamic effects are neglected. Similarly, these functions can be used for the special case of isotropic enrichment functions of moving cracks, as proposed by Belytschko and Chen (2004) and discussed in Section 5.4.3.3.

5.4.3.3 Isotropic crack-tip functions for moving cracks

Belytschko and Chen (2004) used the analytical asymptotic solutions (5.1) and (5.2) near a propagating crack in an isotropic medium to develop the dynamic enrichment functions. A simplified form of these functions can be written as,

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ r_d \cos \frac{\theta_d}{2}, r_s \cos \frac{\theta_s}{2}, r_d \sin \frac{\theta_d}{2}, r_d \sin \frac{\theta_s}{2} \right\} \quad (5.114)$$

where r_d , r_s , θ_d and θ_s are defined in equation (5.13)-(5.16).

Alternatively, Kabiri (2009) proposed the following enrichment functions for dynamic problems based on a simple idea of adding the influence of the crack tip velocity on classical isotropic enrichment functions,

$$\{F_l(r, \theta)\}_{l=1}^4 = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \cos \theta \sin \frac{\theta}{2} \right\} \sqrt{\frac{\gamma_d^2 + \gamma_s^2}{2}} \quad (5.115)$$

with

$$\gamma_d^2 = 1 - \frac{v_c^2}{c_d^2} \sin^2 \theta \quad (5.116)$$

$$\gamma_s^2 = 1 - \frac{v_c^2}{c_s^2} \sin^2 \theta \quad (5.117)$$

Initial results have shown improved dynamic responses, but further investigation is required to assess performance of (5.115) for general dynamic problems. For further details see Kabiri (2009), Rezaei (2010) and Parchei (2012).

5.4.3.4 Propagating enrichments

It is important to change the additional degrees of freedom according to crack propagation. As proposed by Combescure *et al.* (2008) and Rethore *et al.* (2005a) and adopted by Motamedi and Mohammadi (2010a,b, 2012) for dynamic analysis of stationary and moving cracks in orthotropic media, once the crack tip enters an element, new tip enrichment DOFs are assumed for that element. Previous crack tip element contains both set of Heaviside and crack tip DOFs from a previous step, but evaluated from the present crack tip position (Fig. 5.4). As a result, the singularity effect is expected to significantly reduce in previous crack tip element because of its larger distance from the present crack tip position. Several numerical simulations have proved that this method substantially reduces the usual oscillations in the dynamic response which are due to sudden change of degrees of freedom. The method preserves the energy conservation, allowing the acceleration to be reliable for evaluation of J integral and post-processing purposes.

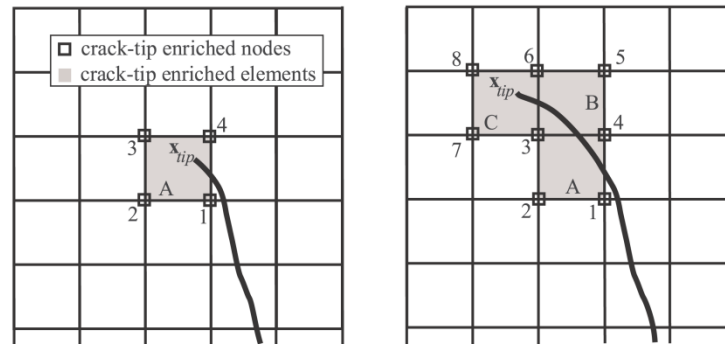


Figure 5.4 Crack tip enriched degrees of freedom for a propagating crack. Enrichments of all tip enriched elements in successive time steps are based on the current position of $\mathbf{x}_{tip}$.

5.4.3.5 Time variations of XFEM enrichment functions

Crack tip enrichment functions for propagating cracks (5.113)-(5.114) are functions of time. As a result, time variations of these functions should be considered in discretising the dynamic equilibrium equation (5.83).

Nevertheless, while many numerical solutions have considered the changing nature of these functions in different time steps, several other simulations have been performed without considering the time derivatives of crack tip enrichment functions, and good results have been reported for most of the cases (Belytschko *et al.*, 2003, Belytschko and Chen, 2004, Rethore *et al.*, 2005a, and Motamedi and Mohammadi, 2010a, 2010b, 2012, etc.). In general, however, it is necessary to account for time derivatives of these functions for problems with fast propagation patterns, and probably in high strain rates.

The crack tip enrichment part of XFEM approximation (5.84), $\mathbf{u}_{tip}^{Enr}(\mathbf{x})$, is in fact a function of time $\mathbf{u}_{tip}^{Enr}(\mathbf{x}, t)$ as it is a function of current position of the crack tip $\mathbf{x}_{tip}$. For a typical crack tip enrichment part of approximation (without shifting),

$$\mathbf{u}_{tip}^{Enr}(\mathbf{x}, t) = \sum_k \phi_k(\mathbf{x}) F(\mathbf{x}, t) \mathbf{b}_k \quad (5.118)$$

the first and second time derivatives of $\mathbf{u}_{tip}^{Enr}(\mathbf{x}, t)$ can be computed from:

$$\dot{\mathbf{u}}_{tip}^{Enr}(\mathbf{x}, t) = \sum_k \phi_k(\mathbf{x}) \dot{F}(\mathbf{x}, t) \mathbf{b}_k(t) + \sum_k \phi_k(\mathbf{x}) F(\mathbf{x}, t) \dot{\mathbf{b}}_k(t) \quad (5.119)$$

$$\begin{aligned} \ddot{\mathbf{u}}_{tip}^{Enr}(\mathbf{x}, t) = & \sum_k \phi_k(\mathbf{x}) \ddot{F}(\mathbf{x}, t) \mathbf{b}_k(t) + \sum_k \phi_k(\mathbf{x}) F(\mathbf{x}, t) \ddot{\mathbf{b}}_k(t) \\ & + 2 \sum_k \phi_k(\mathbf{x}) \dot{F}(\mathbf{x}, t) \dot{\mathbf{b}}_k(t) \end{aligned} \quad (5.120)$$

In order to demonstrate further steps of computation, assume one of the terms of isotropic crack tip enrichment functions (3.31) in crack tip local coordinates, $F(r, \theta) = \sqrt{r} \sin \theta / 2$. Therefore, for a typical point $\mathbf{x}$ with respect to the time dependent crack tip coordinate $\mathbf{x}_{tip}(t)$ (in terms of global coordinates)

$$F(r, \theta) = F(\mathbf{x}, t) = \left\| \mathbf{x} - \mathbf{x}_{tip}(t) \right\|^{\frac{1}{2}} \sin \frac{\theta}{2} \quad (5.121)$$

and its time derivatives become (Menouillard *et al.*, 2010),

$$\dot{F}(\mathbf{x}, t) = -\dot{\mathbf{x}}_{tip}(t) \nabla F(r, \theta) \quad (5.122)$$

$$\ddot{F}(\mathbf{x}, t) = -\ddot{\mathbf{x}}_{tip}(t) \nabla F(r, \theta) - \left\| \dot{\mathbf{x}}_{tip}(t) \right\|^2 \nabla^2 F(r, \theta) \quad (5.123)$$

As a result, Eq. (5.85) is modified to: (Menouillard *et al.*, 2010),

$$\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{K}\mathbf{u}^h = \mathbf{f} - \mathbf{f}_1^b - \mathbf{f}_2^b - \mathbf{f}_3^b \quad (5.124)$$

where new $\mathbf{f}_1^b$, $\mathbf{f}_2^b$ and $\mathbf{f}_3^b$ vectors appear in addition to the original tip enrichment force vector $\mathbf{f}_i^b$, as defined in (5.100),

$$\mathbf{f}_1^b = \left[\int_{\Omega^e} N_i N_j \ddot{F}_\alpha d\Omega \right] \mathbf{b} + \left[\int_{\Omega^e} 2N_i N_j \dot{F}_\alpha d\Omega \right] \dot{\mathbf{b}}, \alpha = 1-4 \quad (5.125)$$

$$\mathbf{f}_2^b = \left[\int_{\Omega} (N_i H_i) (N_j \ddot{F}_{\alpha j}) d\Omega \right] \mathbf{b} + \left[\int_{\Omega} 2(N_i H_i) (N_j \dot{F}_{\alpha j}) d\Omega \right] \dot{\mathbf{b}}, \alpha = 1-4 \quad (5.126)$$

$$\mathbf{f}_3^b = \left[\int_{\Omega} (N_i F_{\alpha i}) (N_j \ddot{F}_{\alpha j}) d\Omega \right] \mathbf{b} + \left[\int_{\Omega} 2(N_i F_{\alpha i}) (N_j \dot{F}_{\alpha j}) d\Omega \right] \dot{\mathbf{b}}, \alpha = 1-4 \quad (5.127)$$

Equations (5.125)-(5.127) can be extended to their shifted forms similar to the procedure used in original definitions (5.99) and (5.100). It is also noted that the derivatives $\dot{F}(\mathbf{x}, t)$ and $\ddot{F}(\mathbf{x}, t)$ vanish when there is no propagation, $\dot{\mathbf{x}}_{tip} = 0$. As a result, Eq. (5.124) reduces to the original form of (5.85).

5.4.4 Time integration schemes

There are a number of time integration techniques that are frequently used in various dynamic problems. They include the explicit and implicit approaches and span a wide range of solutions associated with different complexity, speed, efficiency and accuracy. While the explicit approach is simple and fast, it is less accurate and suffers from stability conditions. On the other hand, implicit approaches are generally more accurate and stable, but they are computationally more expensive. General single time step algorithms are usually based on residual formulations (the time finite element method) or the truncated Taylor series (Rezaei, 2010).

SSpj (single step with approximation of degree p for equations of order j) methods and GNpj (generalized Newmark with degree p and order j) methods generate a wide range of time integration schemes, such as the Euler method, the Crank-Nicolson approach, the Galerkin formulation, the Newmark method and the backward finite difference formulation (Zienkiewicz *et al.*, 2005 and Rezaei, 2010).

Here, only two methods are briefly reviewed. First, the classical Newmark approach is briefly presented and then the concept of time-discontinuity is embedded within an extended finite element method, resulting in a space- time XFEM (STXFEM).

5.4.4.1 Newmark time integration

It is possible to similarly extend the formulation (5.85) to include the effects of velocity based global damping,

$$\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{C}\dot{\mathbf{u}}^h + \mathbf{K}\mathbf{u}^h = \mathbf{f} \quad (5.128)$$

The Newmark time integration scheme can then be adopted to solve for the extended finite element equation of motion at time step n :

$$\mathbf{M}\ddot{\mathbf{u}}_n^h + \mathbf{C}\dot{\mathbf{u}}_n^h + \mathbf{K}\mathbf{u}_n^h = \mathbf{f}_n \quad (5.129)$$

where $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ are the mass, stiffness and damping matrices, respectively, and $\mathbf{f}_n$ is the load vector at the time step n . The final time-discretised simultaneous equations are expressed as:

$$\begin{aligned} \left(\mathbf{M} + \beta\Delta t^2\mathbf{K} + \alpha\Delta t\mathbf{C} \right) \ddot{\mathbf{u}}_n^h = \mathbf{f}_n - \mathbf{K} \left(\mathbf{u}_{n-1}^h + \Delta t\dot{\mathbf{u}}_{n-1}^h + (1-2\beta)\frac{\Delta t^2}{2}\ddot{\mathbf{u}}_{n-1}^h \right) \\ - \mathbf{C} \left(\dot{\mathbf{u}}_{n-1}^h + (1-\alpha)\Delta t\ddot{\mathbf{u}}_{n-1}^h \right) \end{aligned} \quad (5.130)$$

$$\dot{\mathbf{u}}_n^h = \dot{\mathbf{u}}_{n-1}^h + (1-\alpha)\Delta t\ddot{\mathbf{u}}_{n-1}^h + \alpha\Delta t\ddot{\mathbf{u}}_n^h \quad (5.131)$$

$$\mathbf{u}_n^h = \mathbf{u}_{n-1}^h + \Delta t\dot{\mathbf{u}}_{n-1}^h + (1-2\beta)\frac{\Delta t^2}{2}\ddot{\mathbf{u}}_{n-1}^h + \beta\Delta t^2\ddot{\mathbf{u}}_n^h \quad (5.132)$$

where $\mathbf{u}_n$, $\dot{\mathbf{u}}_n$ and $\ddot{\mathbf{u}}_n$ are the global vectors of nodal displacements, nodal velocities and nodal accelerations, respectively. Δt is the time increment at the present time step and the Newmark's parameters are chosen to be $\beta=1/4$ and $\alpha=1/2$ to fulfil the unconditionally stable criterion.

5.4.4.2 Space-time XFEM (STXFEM)

The space-time extended finite element method provides a time integration approximation which guarantees continuity of displacements $\mathbf{u}$ in time, generates linear variations of velocity field $\mathbf{v}$ which can become discontinuous at each time step, and ensures satisfaction of $\dot{\mathbf{u}} = \mathbf{v}$ (Rethore *et al.*, 2005b).

In this section the method proposed by Rethore *et al.* (2005b) is briefly presented. Using the partition of unity concept to approximate the velocity field $\mathbf{v}(t)$ leads to (Rethore *et al.*, 2005 and Chessa and Belytschko, 2004):

$$\mathbf{v}(t) = \sum_{i=0}^N \Lambda_i(t) \mathbf{v}_i^c + \sum_{j=0}^M \sum_{i \in T_j} \Lambda_i(t) H(t-t_j) \mathbf{v}_{i,j}^e \quad (5.133)$$

where $\mathbf{v}_i^c$ are classical velocity degrees of freedom, $\mathbf{v}_{i,j}^e$ are additional degrees of freedom, Λ_j is the time interval shape function, $H(t-t_j)$ is the Heaviside enrichment function to resemble the potential discontinuity in time (see Fig. 5.5) and T_j is the time interval that is

enriched by the enrichment function $H(t-t_j)$. T_j at each time step is limited to only that step and $H(t-t_j)$ vanishes for times before t_j , so Λ_j is defined in the $[t_{j-1}, t_{j+1}]$ time interval. Therefore, within the time interval $I_{n+1} =]t_n; t_{n+1}[$, only the Heaviside enrichment $H(t-t_n)$ remains present.

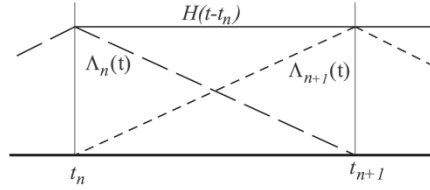


Figure 5.5 Linear shape functions and the Heaviside function within the time interval I_{n+1} (Rethore *et al.*, 2005b).

Velocity and displacement fields can now be defined from their continuum and enriched parts,

$$\mathbf{v}(t) = \mathbf{v}^c(t) + \mathbf{v}^e(t) \quad (5.134)$$

$$\mathbf{u}(t) = \mathbf{u}^c(t) + \mathbf{u}^e(t) \quad (5.135)$$

where

$$\mathbf{v}^c(t) = \mathbf{v}_n^c \Lambda_n(t) + \mathbf{v}_{n+1}^c \Lambda_{n+1}(t) \quad (5.136)$$

$$\mathbf{v}^e(t) = \mathbf{v}_{n+1}^e \Lambda_n(t) H(t-t_n) \quad (5.137)$$

and

$$\Lambda_n(t) = \frac{t_{n+1} - t}{t_{n+1} - t_n} \quad (5.138)$$

$$\Lambda_{n+1}(t) = \frac{t - t_n}{t_{n+1} - t_n} \quad (5.139)$$

Intentionally, $\mathbf{v}_{n,n}^e$ has been replaced by $\mathbf{v}_{n+1}^e$ to insist that it has to be evaluated from the solution in I_{n+1} time step.

In order to satisfy the conditions $\dot{\mathbf{u}} = \mathbf{v}$ and $\mathbf{u}(t_n^+) = \mathbf{u}(t_n^-)$ at each time step, the displacement field is obtained as,

$$\mathbf{u}^c(t) = \mathbf{u}_n^c + \int_{t_n}^t \mathbf{v}^c(\tau) d\tau \quad (5.140)$$

$$\mathbf{u}^e(t) = \mathbf{u}_n^e + \int_{t_n}^t \mathbf{v}^e(\tau) d\tau \quad (5.141)$$

and the weighted residual form of the equation of motion in time step I_{n+1} can be written as,

$$\int_{t_n^+}^{t_{n+1}^-} w(t) [\mathbf{m}\dot{\mathbf{v}}(t) + \mathbf{k}\mathbf{u}(t)] dt + w(t_n^+) \mathbf{m}\mathbf{v}_{n+1}^e = \int_{t_n^+}^{t_{n+1}^-} w(t) \mathbf{f}(t) dt \quad (5.142)$$

Two independent set of weightings have to be considered to account for additional degrees of freedom. The discrete form of (5.142) can be written as (Rethore *et al.*, 2005b),

$$\mathbf{\Pi}_1 \bar{\mathbf{u}}_{n+1} = \mathbf{\Pi}_0 \bar{\mathbf{u}}_n + \mathbf{F}_{n+1} \quad (5.143)$$

$$\bar{\mathbf{u}}_n = \begin{bmatrix} \mathbf{u}_n^c \\ \mathbf{u}_n^e \\ \mathbf{v}_n^c \\ \mathbf{v}_n^e \end{bmatrix} \quad (5.144)$$

$$\mathbf{\Pi}_0 = \begin{bmatrix} 1 & 0 & \frac{\Delta t}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\Delta t \mathbf{k} & -\Delta t \mathbf{k} & \mathbf{m} + (\beta_1 - \gamma_1) \Delta t^2 \mathbf{k} & 0 \\ -\Delta t \mathbf{k} & -\Delta t \mathbf{k} & \mathbf{m} + (\beta_2 - \gamma_2) \Delta t^2 \mathbf{k} & 0 \end{bmatrix} \quad (5.145)$$

$$\mathbf{\Pi}_1 = \begin{bmatrix} 1 & 0 & -\frac{\Delta t}{2} & 0 \\ 0 & 1 & 0 & -\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{m} + \beta_1 \Delta t^2 \mathbf{k} & (\delta_1 - 1) \mathbf{m} + (\gamma_1 - \beta_1) \Delta t^2 \mathbf{k} \\ 0 & 0 & \mathbf{m} + \beta_2 \Delta t^2 \mathbf{k} & (\delta_2 - 1) \mathbf{m} + (\gamma_2 - \beta_2) \Delta t^2 \mathbf{k} \end{bmatrix} \quad (5.146)$$

$$\mathbf{F}_{n+1} = \begin{bmatrix} 0 \\ 0 \\ \Delta t [(1 - \gamma_1) \mathbf{f}_n + \gamma_1 \mathbf{f}_{n+1}] \\ \Delta t [(1 - \gamma_2) \mathbf{f}_n + \gamma_2 \mathbf{f}_{n+1}] \end{bmatrix} \quad (5.147)$$

and

$$2\beta_i \frac{1}{\Delta t} \int_{t_n^+}^{t_{n+1}^-} w_i(t) dt = \frac{1}{\Delta t^3} \int_{t_n^+}^{t_{n+1}^-} w_i(t) (t - t_n^+)^2 dt \quad (5.148)$$

$$\gamma_i \frac{1}{\Delta t} \int_{t_n^+}^{t_{n+1}^-} w_i(t) dt = \frac{1}{\Delta t^2} \int_{t_n^+}^{t_{n+1}^-} w_i(t)(t - t_n^+) dt \quad (5.149)$$

$$\delta_i \frac{1}{\Delta t} \int_{t_n^+}^{t_{n+1}^-} w_i(t) dt = w_i(t_n^+) \quad (5.150)$$

The solution procedure consists of evaluation of unknown variables $\mathbf{v}_{n+1}^e, \mathbf{v}_{n+1}^c, \mathbf{u}_{n+1}^e$ and $\mathbf{u}_{n+1}^c$ from the known values of $\mathbf{v}_n^e, \mathbf{v}_n^c, \mathbf{u}_n^e$ and $\mathbf{u}_n^c$ at each time step. Six parameters $\{\beta_i, \gamma_i, \delta_i, i=1,2\}$ are determined from Eqs. (5.148)-(5.150), which need proper selection of weight functions $w_i, i=1,2$ to achieve the optimal solution (such as (5.152) and (5.153)).

The constant piecewise acceleration within the time interval I_{n+1} is obtained from the derivative of (5.140) and (5.141)

$$\mathbf{a}(t) = \mathbf{a}_{n+1} = \frac{1}{\Delta t} (\mathbf{v}_{n+1}^c - \mathbf{v}_{n+1}^e - \mathbf{v}_n^c) \quad (5.151)$$

STXFEM can be degenerated to the discontinuous Galerkin time integration scheme (Li *et al.*, 2003) by setting,

$$w_1 = \lambda_n \quad (5.152)$$

$$w_2 = \lambda_{n+1} \quad (5.153)$$

where λ_n and λ_{n+1} are determined from (5.138) and (5.139). The resulting $\{\beta_i, \gamma_i, \delta_i, i=1,2\}$ coefficients are:

$$\beta_1 = \frac{1}{12}, \beta_2 = \frac{1}{4}, \gamma_1 = \frac{1}{3}, \gamma_2 = \frac{2}{3}, \delta_1 = 2, \delta_2 = 2 \quad (5.154)$$

Rethore *et al.* (2005b) have computed the spectral radius of the STXFEM approach for an un-damped SDOF system subjected to an initial velocity. They have illustrated that the spectral radius remains always less than 1, indicating that the method is unconditionally stable. Similar conclusions have been made by Sukumar and Prevost (2003) and Hughes (2000).

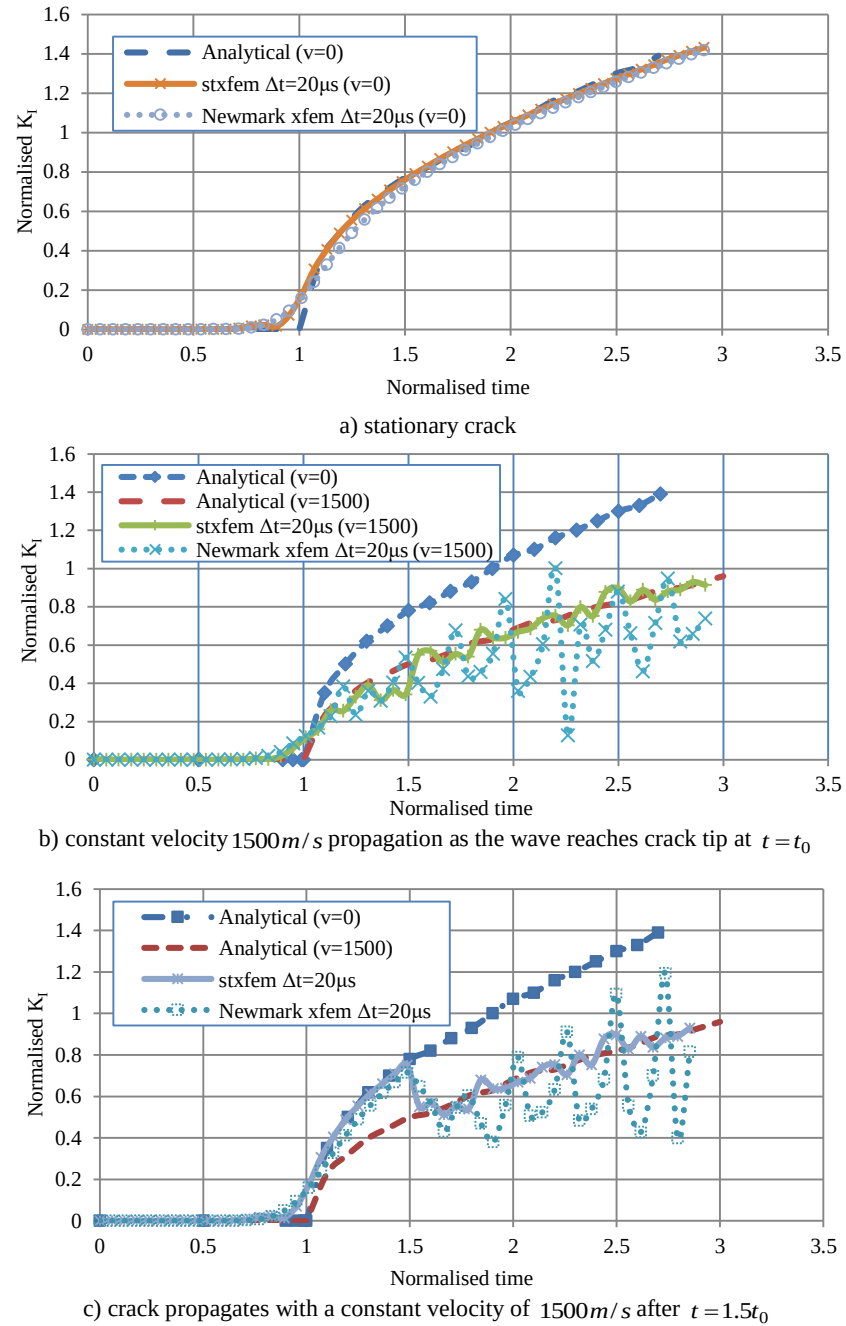


Figure 5.18 Comparison of normalised dynamic stress intensity factor predicted by STXFEM and Newmark XFEM for a) stationary crack ($v_c = 0$), b) crack propagates with a constant $v_c = 1500$ m/s as soon as the wave reaches its tip at $t = t_0$, c) crack propagates with a constant velocity of $v_c = 1500$ m/s after $t = 1.5t_0$ (Kabiri, 2009 and Rezaei, 2010).

